Int. J. Solids Structures Vol. 26, No. 3, pp. 369–373, 1990
Printed in Great Britain.

0020–7683/90 $3.00 + .00

ON THE FINITE ELASTOSTATIC DEFORMATION OF THIN-WALLED SPHERES AND CYLINDERS

A. J. Willson and P. J. Myers†
Department of Mathematics, University of Leicester, University Road, Leicester LEI 7RH, U.K.

(*Received* 28 *March* 1989)

Abstract—The finite static deformations of thin-walled spheres and cylinders composed of general compressible isotropic elastic material, under the action of applied steady internal pressure, are investigated theoretically. The relation between applied pressure and deformed radius is obtained in a simple parametric form. The results of Chung *et al.* (1986, *Int. J. Solids Structures* **22**, 1557–1570) are recovered for the model proposed by Blatz and Ko (1962, *Trans. Soc. Rheol.* **6**, 223–251) for polyurethane foam rubber, and the analysis is then applied numerically to the generalized Blatz–Ko model discussed by Willson and Myers (1988, *Int. J. Engng Sci.* **26**, 509–517) and by Myers (1988, Ph.D. Thesis, University of Leicester, U.K.).

1. INTRODUCTION

The elastic deformations of hollow spheres and cylinders under the action of imposed steady internal pressures have been investigated by several workers. In order to facilitate calculation, however, particular forms for the elastic strain-energy function were assumed. For example, in a recent study by Chung *et al.* (1986), a detailed investigation was made for spheres and cylinders composed of an elastic foam rubber material described by a strain-energy function proposed by Blatz and Ko (1962). In the present note we show that for spheres and cylinders with sufficiently thin walls the pressure–radius relation can be derived in a simple way without reference to the specific strain-energy function. It is then shown that for polyurethane foam rubber the results of Chung *et al.* (1986) are recovered, and for illustrative purposes our results are then applied to the generalized Blatz–Ko model discussed by Willson and Myers (1988) and investigated in detail by Myers (1988).

2. THE PRESSURIZED HOLLOW CYLINDER

Suppose that in the undeformed configuration the cylinder occupies the region $a < r < b$. When a steady internal pressure p is applied at the inner wall $r = a$, the resulting deformation carries the particle originally at (r, θ, z) to the site (R, Θ, Z). We assume that the deformation is one of axisymmetric plane strain so that

$$R = R(r), \quad \Theta = \theta, \quad Z = z, \tag{1}$$

where $R(r)$ is some function whose form is yet to be determined. Then the principal stretches λ_i $(i = 1, 2, 3)$ are given by

$$\lambda_1 = \mathrm{d}R/\mathrm{d}r, \quad \lambda_2 = R/r, \quad \lambda_3 = 1. \tag{2}$$

If the strain-energy per unit undeformed volume is denoted by $W(\lambda_1, \lambda_2, \lambda_3)$, then the corresponding principal stresses τ_i are given by

† Present address: Computer Centre, University of Leicester, University Road, Leicester LEI 7RH, U.K.

$$\tau_i = \frac{\lambda_i}{\lambda_1\lambda_2\lambda_3}\frac{\partial W}{\partial \lambda_i} \quad \text{(no summation implied).} \tag{3}$$

The true stress tensor τ is diagonal with elements τ_{RR}, $\tau_{\Theta\Theta}$, τ_{ZZ}. For equilibrium, with zero body-force, we have

$$\frac{d\tau_{RR}}{dr} + \frac{1}{R}\frac{dR}{dr}(\tau_{RR} - \tau_{\Theta\Theta}) = 0. \tag{4}$$

Put $\varepsilon = (b-a)/a$, $u = (r-a)/(a\varepsilon)$, so that within the material u runs from 0 to 1, and for a thin-walled cylinder $\varepsilon \ll 1$. We assume that $R(r)$ can be expanded thus

$$R(r) = A[1 + \alpha\varepsilon u + O(\varepsilon u)^2] \tag{5}$$

where A, α are independent of r, and we assume that α is $O(1)$ with respect to ε.

Suppose further that the internal pressure $p = \gamma\varepsilon$, where γ is $O(1)$. Since

$$\tau_{RR} = \begin{cases} -\gamma\varepsilon & \text{when } r = a \\ 0 & \text{when } r = b \end{cases} \tag{6}$$

then

$$\frac{d\tau_{RR}}{dr} = \frac{\gamma}{a} + O(\varepsilon).$$

Also

$$\frac{1}{R}\frac{dR}{dr} = \frac{\alpha}{a} + O(\varepsilon).$$

Further, $\tau_{RR} = O(\varepsilon)$ throughout the material, so that the equation of equilibrium requires

$$\tau_{\Theta\Theta} = \frac{\gamma}{\alpha} + O(\varepsilon). \tag{7}$$

Retaining only the leading terms in the approximation, we have $\tau_{RR} = 0$, $\tau_{\Theta\Theta} = \gamma/\alpha$ when

$$\lambda_1 = A\alpha/a, \quad \lambda_2 = A/a, \quad \lambda_3 = 1. \tag{8}$$

Replacing τ_{RR}, $\tau_{\Theta\Theta}$ by their expressions in terms of W and the λ_i, we see that the final result for thin-walled cylinders, valid for general compressible isotropic elastic materials, can be expressed as the pair of equations

$$\frac{\partial W}{\partial \lambda_1} = 0, \quad \gamma = \frac{a}{A}\frac{\partial W}{\partial \lambda_2}, \tag{9}$$

with λ_1, λ_2, λ_3 given by (8), the variable α then serving as a parameter for the pressure–radius relation.

For polyurethane foam rubber the strain-energy function proposed by Blatz and Ko (1962) is given by

$$W = (\mu/2)(2\lambda_1\lambda_2\lambda_3 + \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 5), \tag{10}$$

so that (9) yields

$$A/a = \alpha^{-3/4}, \quad \gamma/\mu = \alpha - \alpha^3, \tag{11}$$

in accordance with Chung *et al.* (1986).

3. THE PRESSURIZED HOLLOW SPHERE

We now apply a similar analysis to the case of a hollow sphere. We work in spherical polar coordinates so that the particle at (r, θ, ϕ) in the undeformed state moves to the site (R, Θ, Φ) after deformation, where

$$R = R(r), \quad \Theta = \theta, \quad \Phi = \phi. \tag{12}$$

The principal stretches λ_i are now given by

$$\lambda_1 = \mathrm{d}R/\mathrm{d}r, \quad \lambda_2 = \lambda_3 = R/r. \tag{13}$$

The equation of equilibrium can be written as

$$\frac{\mathrm{d}\tau_{RR}}{\mathrm{d}r} + \frac{2}{R}\frac{\mathrm{d}R}{\mathrm{d}r}(\tau_{RR} - \tau_{\Theta\Theta}) = 0 \tag{14}$$

and it is readily found that in the case of a hollow thin-walled sphere our result, again valid for general compressible isotropic elastic materials, is the pair of equations

$$\frac{\partial W}{\partial \lambda_1} = 0, \quad \gamma = 2\left(\frac{a}{A}\right)^2 \frac{\partial W}{\partial \lambda_2}, \tag{15}$$

where

$$\lambda_1 = A\alpha/a, \quad \lambda_2 = \lambda_3 = A/a. \tag{16}$$

For polyurethane foam rubber governed by (10), this leads to

$$A/a = \alpha^{-3/5}, \quad \gamma/\mu = 2(\alpha - a^3), \tag{17}$$

again in agreement with Chung *et al.* (1986).

In a recent review, Beatty (1987) has discussed the theory and application of finite elasticity and has given similar results for the thin-walled sphere composed of certain Blatz–Ko materials.

4. THE GENERALIZED BLATZ–KO MODEL

To illustrate the results (9) for the cylinder and (15) and for the sphere we consider the generalized Blatz–Ko model investigated by Willson and Myers (1988) in which

$$W = \mu\left[\lambda_1\lambda_2\lambda_3 + \frac{n}{2}(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}) + \frac{(n-1)}{2}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) + \tfrac{1}{2} - 3n\right], \tag{18}$$

where n is some constant. It will be noted that (18) reduces to (10) in the special case $n = 1$. Willson and Myers (1988) and Myers (1988) have advanced reasons for imposing the constraint $n \geqslant 1$ upon (18) so as to secure a physical response to various stress-systems which is in accordance with our natural expectations.

Then for the thin-walled cylinder, using (9) and (18), we have

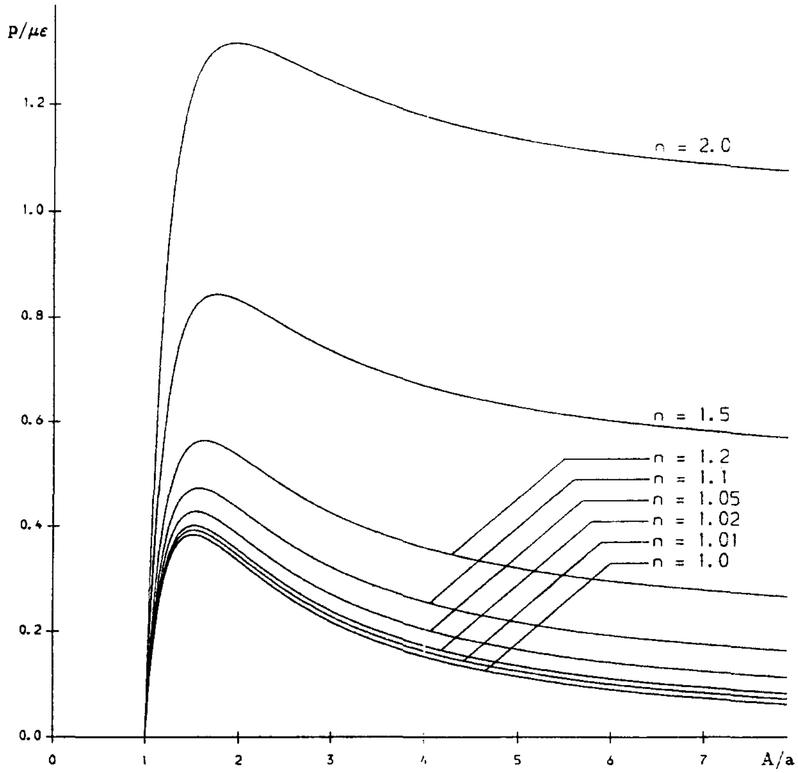

Fig. 1. The relation between applied pressure and deformed radius (both in dimensionless form) for thin-walled cylinders composed of generalized Blatz–Ko materials, for the values of n shown.

$$\frac{A}{a} = \left[\frac{n}{\alpha^3+(n-1)\alpha^4}\right]^{1/4}, \quad \frac{\gamma}{\mu} = (1-\alpha^2)[\alpha+(n-1)(1+\alpha^2)]. \tag{19}$$

In Fig. 1 we show the relation between pressure p and deformed radius A for various values of n. For each value of n the pressure increases from zero, reaches a maximum and then decreases, so that $p/\mu\varepsilon$ tends to $(n-1)$ as $A/a \to \infty$. At this pressure maximum the value of A/a increases only slowly with n but the maximum pressure itself is sensitive to changes in n. Of course the curve for $n = 1$ depicts the result (11) for polyurethane foam rubber.

For the thin-walled sphere, (15) and (18) yield

$$\left(\frac{A}{a}\right)^5 - \frac{n}{\alpha^3} + (n-1)\left(\frac{A}{a}\right)^4 \alpha = 0 \tag{20}$$

and

$$\frac{\gamma}{\mu} = 2\left[\alpha+(n-1)\left(\frac{a}{A}\right)-n\left(\frac{a}{A}\right)^5\right]. \tag{21}$$

Putting $A\alpha/a = s$, a new parameter, we obtain the simpler forms

$$\frac{A}{a} = \left[\frac{n-(n-1)s^4}{s^3}\right]^{1/2}, \quad \frac{\gamma}{\mu} = 2\left(\frac{a}{A}\right)\left[s+n-1-n\left(\frac{a}{A}\right)^4\right]. \tag{22}$$

In Fig. 2 the pressure–radius relation is shown, again for various values of n. Along any one curve, corresponding to a particular choice of n, the parameter s starts from the value unity and decreases to zero. The pressure increases from zero, attains a maximum, and this time tends to zero as $A/a \to \infty$. The curve $n = 1$ corresponds to the result (17).

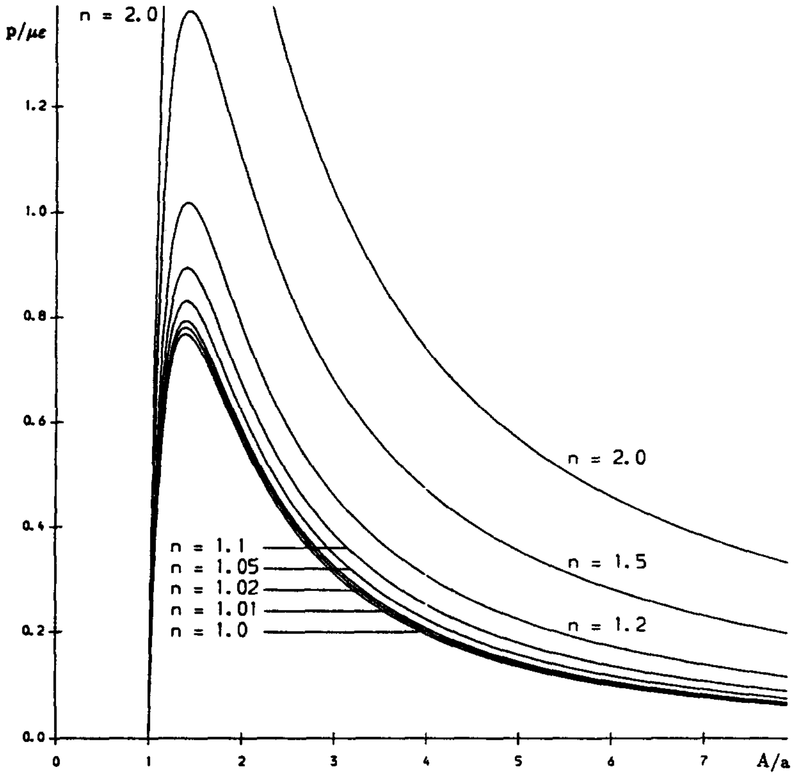

Fig. 2. The relation between applied pressure and deformed radius (both in dimensionless form) for thin-walled spheres composed of generalized Blatz–Ko materials, for the values of n shown.

In summary, the results (9), (15) obtained here afford especially simple parametrized expressions for the pressure–radius relation for thin-walled cylinders and spheres under the action of steady internal pressure.

REFERENCES

Beatty, M. F. (1987). Topics in finite elasticity: hyperelasticity of rubber, elastomers, and biological tissues—with examples. *Appl. Mech. Rev.* **40**(12), 1699–1734.

Blatz, P. J. and Ko, W. L. (1962). Application of finite elastic theory to the deformation of rubbery materials. *Trans. Soc. Rheol.* **6**, 223–251.

Chung, D.-T., Horgan, C. O. and Abeyaratne, R. (1986). The finite deformation of internally pressurized hollow cylinders and spheres for a class of compressible elastic materials. *Int. J. Solids Structures* **22**(12), 1557–1570.

Myers, P. J. (1988). The influence of a primary stress upon the propagation of small-amplitude elastic disturbances. Ph.D Thesis, University of Leicester, U.K.

Willson, A. J. and Myers, P. J. (1988). A generalization of Ko's strain-energy function. *Int. J. Engng Sci.* **26**, 509–517.